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An Approximation Theorem for Topological Lambda Models and the Topological Incompleteness of Lambda Calculus*

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It is well known that a reflexive object in the Cartesian closed category of complete partial orders and Scott-continuous functions is a model of λ-calculus (briefly a topological model). A topological model, through the interpretation function, induces a λ-theory, i.e., a congruence relation on λ-terms closed under α- and β-reduction. It is natural to ask if all possible λ-theories are induced by a topological model, i.e., if topological models are complete w.r.t. λ-calculus. The authors prove an Approximation Theorem, which holds in all topological models. Using this theorem, they analyze some topological models and their induced λ-theories, and they exhibit a λ-theory which cannot be induced by a topological model. So they prove that topological models are not complete w.r.t. λ-calculus.

1. INTRODUCTION

It is well known that a reflexive object in the Cartesian closed category of complete partial orders (CPOs) and Scott-continuous functions is a model of the λ-calculus. D is a reflexive object if $[D \to D]$, the CPO of all Scott-continuous functions from D to D, is a retract of D; i.e., there are continuous maps $F: D \to [D \to D]$ and $G: [D \to D] \to D$ such that $F \cdot G = \mathrm{Id}_{[D \to D]}$. In this paper we shall deal with this kind of model of the λ-calculus. These models are referred to in the literature as continuously complete λ-models, since all continuous functions defined on them are representable.We shall refer to them more simply as topological models. They provide, through the interpretation function, a geometrical means to define a λ-theory (i.e., a congruence relation on λ-terms closed under α- and β-reduction).

Most of the past work on topological models of the λ-calculus has focused on sensible models (i.e., models where all unsolvable terms are equated). However, non-sensible models are also interesting, since, giving non-trivial computational meaning to a wider class of λ-terms, they suggest operational properties of terms independent from a head reduction-type termination property.

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Up to now no general methodology has been known for studying topological models and the lambda theories induced by them.

Two natural questions arise:

(i) Are there special equations, true in all topological models, which are not derivable by $\lambda-\beta$-conversion?

(ii) Are all possible λ-theories induced by a topological model?

Question (ii) could be rephrased more classically as follows: are topological models complete w.r.t. λ-calculus, in the sense that given a set of equations between λ-terms either this set is inconsistent or there exists a topological λ-model where this set of equations is satisfied?

Question (i) is still open. In Section 5 of this paper we exhibit a λ-theory which cannot be induced by a topological model, thus giving a negative answer to question (ii).

We now describe in more detail the content of this paper.

In Section 2 we provide a major tool in the analysis of the λ-theory induced by a topological model: an approximation theorem. Namely, we extend to all topological models of λ-calculus arising from an inverse limit construction in the ccc of CPOs the methodology of indexed reduction, introduced by Hyland and Wadsworth [4, 12, 13]. Next we consider an extension Λ^* of the λ-calculus, obtained introducing a new constant Φ, and we prove that the interpretation of a term of pure λ-calculus is the supremum of the interpretations of normal forms of Λ^*. Moreover, we particularize our approximation theorem to models of $\lambda-\beta-\eta$ calculus, in view of our further applications.

In Sections 3 and 4 we use this approximation theorem to study two particular topological models, which induce non-sensible theories. One of these is the model proposed by Park [8]. We prove that the theory induced by this model is non-semisensible; in particular, it equates any solvable term to an unsolvable one.

In Section 5 we first introduce the notion of contextual theory, which turns out to be particularly significant in making apparent the operational behaviour of terms. We then prove an unpublished result of D. Scott which shows that every λ-theory has a contextual definition, thereby giving an operational meaning to the notion of λ-theory. In order to analyze, from the operational point of view, the theories induced by the models defined in the previous sections, we prove that each one is strictly included in a particular contextual theory, which arises from syntactically signficant contextual considerations. Finally we prove that there is no consistently complete topological model of the λ-calculus which induces the contextual theory which strictly includes the theory of Park's model.

In the last section we give two semisensible models in which the constant Φ is not λ-definable. A specific property of these two models leads us to refine the approximation theorem so as to obtain λ-definable approximants. We show that the classical approximation theorem for the Scott D_∞-model can be obtained similarly.

2. The Approximation Theorem

We assume that the reader is familiar with λ-calculus, in particular with D_∞-models (see [1, 4, 10, 12]).

Let D^0 be a CPO, i.e., a partially ordered set with a least element, where all directed sets have a least upper bound. We denote by $D_{(i,j)} = \varprojlim D^n_{(i,j)}$ the inverse limit space built from D^0 with (i, j) as the initial projection system between D^0 and $[D^0 \to D^0]$ (i.e., $i: D^0 \to [D^0 \to D^0]$, $j: [D^0 \to D^0] \to D^0$ are such that $i \circ j \sqsubseteq \mathrm{Id}_{[D^0 \to D^0]}$, $j \circ i = \mathrm{Id}_{D^0}$, where Id_D is the identity function on D). We omit the subscript (i, j) when it is clear from the context. We always identify $D^n_{(i,j)}$ with the corresponding subdomain of $D_{(i,j)}$.

Given $D_{(i,j)}$ and $x \in D_{(i,j)}$, we denote by x_n, $\perp_n$, and $\top_n$, respectively, the projection of x on $D^n_{(i,j)}$ and the bottom element and the top element of $D^n_{(i,j)}$, when it exists.

Let Λ be the set of terms of λ-calculus, built on a set Var of variables. The interpretation of $M \in \Lambda$ in $D_{(i,j)}$, in an environment ξ: Var $\to D_{(i,j)}$, is inductively defined as

$$[\![x]\!]^{D(i,j)}_\xi = \xi(x)$$
$$[\![MN]\!]^{D(i,j)}_\xi = [\![M]\!]^{D(i,j)}_\xi \cdot [\![N]\!]^{D(i,j)}_\xi$$
$$[\![\lambda x . M]\!]^{D(i,j)}_\xi = \lambda d : D_{(i,j)} \cdot [\![M]\!]^{D(i,j)}_{\xi[x/d]}.$$

Given any CPO L and $x, y \in L$, we denote by $C_x: L \to L$ the function $\lambda z \in L . x$, and by $f_{x,y}: L \to L$ the function $\lambda z \in L$. if $z \sqsupseteq x$ then y else $\perp$.

In order to study the standard D_∞ model, Hyland [4] and Wadsworth [12, 13] introduced the concepts of indexed term and indexed reduction. We extend these notions to any $D_{(i,j)}$, thus obtaining a general approximation theorem.

Let Λ be the set of λ-terms. Let Λ^* be the language obtained from Λ by adjoining a special constant denoted by Φ; i.e., the terms of Λ^* are built up from variables and Φ by application and abstraction. The value of a term of Λ^* in $D_{(i,j)}$ is obtained by adjoining, to the previously defined semantic clauses, the following one:

$$[\![\Phi]\!]^{D(i,j)} = ([\![I]\!]^{D(i,j)})_1$$

(where $I = \lambda x . x$).

Definition 1. (i) An *indexed term* (M, I) is a term $M \in \Lambda^*$, together with a map I from the subterms of M to the natural numbers.

(ii) The value $[\![M^I]\!]^{D(i,j)}$ of (M, I) is given by

$$[\![x^I]\!]^{D(i,j)}_\xi = (\xi(x))_{I(x)}$$
$$[\![(MN)^I]\!]^{D(i,j)}_\xi = (([\![M^I]\!]^{D(i,j)}_\xi \cdot ([\![N^I]\!]^{D(i,j)}_\xi))_{I(MN)}$$
$$[\![(\lambda x . M)^I]\!]^{D(i,j)}_\xi = (\lambda d . [\![M^I]\!]^{D(i,j)}_{\xi[x^{I(x)}/d]})_{I(\lambda x . M)}.$$

Informally, an indexed term is a term with an integer associated to every subterm, and the intended meaning is that the corresponding projection of the value of the subterm is to be taken.

DEFINITION 2. The *indexed reduction rules* are defined as follows:

$$(\beta_I) \quad ((\lambda x.P^n)^{m+1} Q^p)^h \xrightarrow[\beta_I]{} (P[x/Q^a])^b \ ^1$$

where $b = \min(n, m, h)$, $a = \min(m, p, q)$, where q is the index of the substituted occurrence of x;

$$(\beta_0) \quad (\lambda x.P^n)^0 Q \xrightarrow[\beta_0]{} \Phi(\lambda x.P^n)^0 Q.$$

LEMMA 1. *$M \in \Lambda$. $[\![M]\!]_\xi^{D(i,j)} = \bigsqcup \{[\![M']\!]_\xi^{D(i,j)} \mid (M, I)$ is an indexed term$\}$.*

Proof. By structural induction (see [13]). ∎

LEMMA 2. *The indexed reduction rules preserve the value of an indexed term.*

Proof. Easy, by induction on the definition of the value of an indexed term (see [13]). ∎

DEFINITION 3. Let (M, I) be an indexed term.

(i) The *degree* of a redex $(\lambda x.P)^{m+1} Q^P$ in (M, I) is $m+1$.

(ii) (M, I) is in weak normal form (wnf) iff it does not have occurrences of (β_I)-redexes; i.e., for every redex $(\lambda x.P)Q$ occurring in M, $I(\lambda x.P) = 0$.

Our definition of indexed reduction differs from the classical one [12] only in the β_0-reduction rule. In the classical case every indexed λ-term is strongly normalizing [1]; i.e., every reduction path reaches the normal form (*nf*). Using a similar argument, it is straightforward to prove:

LEMMA 3. *Let (M, I) be an indexed term. Then (M, I) strongly reduces to a wnf.*

LEMMA 4. *$M \in \Lambda$. $[\![M]\!]_\xi^{D(i,j)} = \bigsqcup \{[\![\bar{M}']\!]_\xi^{D(i,j)} \mid (M, J)$ is an indexed term and $(\bar{M}, I)$ is the wnf of $(M, J)\}$.*

Proof. By Lemmas 1, 2, and 3. ∎

LEMMA 5. *Any indexed term is strongly normalizing.*

Proof. The β_0-reduction rule does not generate new redexes. Hence the proof follows from Lemma 3. ∎

[1] Let R be a reduction rule. $A \xrightarrow[R]{} B$ indicates that A R-reduces to B in exactly 1 step, while $A \xrightarrow[R]{}\!\!\!\!\rightarrow B$ indicates that A R-reduces to B in a number $n \geq 0$ of steps.

DEFINITION 4. $M \in \Lambda$.

(i) The *direct approximant* of M is the nf $A \in \Lambda^*$ obtained from M by replacing each redex $(\lambda x.P)Q$ by $\Phi(\lambda x.P)Q$.

(ii) The set of *approximants* of M is the set

$$\mathcal{A}(M) = \{A \mid \exists M', M \twoheadrightarrow_{\beta} M' \text{ and } A \text{ is the direct approximant of } M\}.$$

Now, we are able to prove the Approximation Theorem, which, of course, is semantically significant only in the case $[\![I^1]\!]^{D(i,j)} \sqsubsetneq [\![I]\!]^{D(i,j)}$.

APPROXIMATION THEOREM (I).

$$[\![M]\!]_{\xi}^{D(i,j)} = \bigsqcup \{[\![A]\!]_{\xi}^{D(i,j)} \mid A \in \mathcal{A}(M)\}.$$

Proof. From Lemmas 4 and 5, $[\![M]\!]_{\xi}^{D(i,j)} = \bigsqcup \{[\![\bar{M}^I]\!]_{\xi}^{D(i,j)} \mid (M, J)$ is an indexed term, and $(\bar{M}, I)$ is the normal form of $(M, J)\}$.

On the other hand, for every $A \in \mathcal{A}(M)$, $[\![A]\!]_{\xi}^{D(i,j)} \sqsubseteq [\![M]\!]_{\xi}^{D(i,j)}$.

Then the thesis follows, since erasing the indices from $(\bar{M}, I)$, we obtain exactly the approximants of M. ∎

It is worth noting that the above results generalize straightforwardly to topological models D arising from inverse limit constructions, which are not necessarily isomorphic to their own function spaces. What is actually needed is that the function space $[D^n \to D^n]$ is a finitary projection of D^{n+1}.

We can syntactically characterize the set $\mathcal{A}$ of approximants as follows:

DEFINITION 5. (i) $\mathcal{A}$ is inductively defined as follows:

$x \in \mathcal{A}$ for every variable x

$\Phi \in \mathcal{A}$

$A_i \in \mathcal{A}$ $(1 \leqslant i \leqslant m) \Rightarrow \forall n, \forall \zeta$ variable, $\lambda x_1 \ldots x_n . \zeta A_1 \ldots A_m \in \mathcal{A}$

and $\lambda x_1 \ldots x_n . \Phi(\lambda\zeta . A_1)A_2 \ldots A_m \in \mathcal{A}$.

(ii) Let $A = \lambda x_1 \ldots x_n . \zeta A_1 \ldots A_m \in \mathcal{A}$. We call A a head normal form (*hnf*) in the case where ζ is a variable, a Φ-head normal form in the case $\zeta \equiv \Phi$.

COROLLARY 1. *Let* $M \in \Lambda$.

(i) *M is an nf* $\Leftrightarrow \mathcal{A}(M) = \{M\}$.

(ii) *M is an hnf* ($M \equiv \lambda x_1 \ldots x_n . \zeta M_1 \ldots M_m$ *where* $\zeta \in \mathrm{Var}$) $\Leftrightarrow \mathcal{A}(M) = \{\lambda x_1 \ldots x_n . \zeta A_1 \ldots A_m \mid A_i \in \mathcal{A}(M_i)\}$.

(iii) *M has an hnf* $\Leftrightarrow \exists A$, $A \in \mathcal{A}(M)$, *A is an hnf but not a Φ-head nf*

(iv) *M is unsolvable* $\Leftrightarrow \forall A \in \mathcal{A}(M)$, *$A$ is a Φ-head nf.*

Proof. (i), (ii), (iii) follow directly from the definition of $\mathcal{A}(M)$. For (iv) remember that an unsolvable term and all its reducts are of the shape

$$\lambda x_1 \dots x_n . R\, M_1 \dots M_m,$$

where R is a redex. Then, if $A \in \mathcal{A}(M)$, since by definition A must be in normal form, A is a Φ-head nf. ∎

We now prove that the set of the interpretations in $D_{(i,j)}$ of the approximants of a term is a directed set. First we need a lemma, which will be extensively used also in the following sections. Note that here terms are partially indexed. The interpretation of a subterm indexed by n is the projection onto $D^n_{(i,j)}$ of the interpretation of the subterm in $D_{(i,j)}$.

LEMMA 6. $[\![(\lambda x.M)^0 N]\!]_\xi^{D(i,j)} \sqsubseteq [\![(\lambda x.M)N^0]\!]_\xi^{D(i,j)}$.

Proof.

$$\begin{aligned}[\![(\lambda x.M)^0 N]\!]_\xi^{D(i,j)} &\sqsubseteq [\![(\lambda x.M)^1 N]\!]_\xi^{D(i,j)} \\ &= [\![(M[x/N^0])^0]\!]_\xi^{D(i,j)} \sqsubseteq [\![M[x/N^0]]\!]_\xi^{D(i,j)}). \quad ∎\end{aligned}$$

LEMMA 7. $M \xrightarrow[\beta]{} M' \Rightarrow \forall \xi.\ [\![A]\!]_\xi^{D(i,j)} \sqsubseteq [\![A']\!]_\xi^{D(i,j)}$, *where A and A' are the direct approximants respectively of M and M'.*

Proof. Let $M = C[(\lambda x.P)Q]$[2] and $N = C[P[x/Q]]$, and let P' and Q' be respectively the direct approximants of P and Q.

Then $[\![A]\!]_\xi^{D(i,j)} = [\![C'[\Phi(\lambda x.P')Q']]\!]_\xi^{D(i,j)}$ and $[\![C'[\Phi(\lambda x.P')Q']\!]_\xi^{D(i,j)} = [\![C'[(\lambda x.P')^0 Q']\!]_\xi^{D(i,j)}$.

Let R be the direct approximant of $P[x/Q]$. If Q is not an abstraction, $R = P'[x/Q']$. Otherwise, let us denote by x^{f} and x^{a} the occurrences of x in P respectively in functional and argument position; by definition of direct approximant, $R = P'[x^{\mathrm{f}}/\Phi Q', x^{\mathrm{a}}/Q'] = P'[x^{\mathrm{f}}/Q'^0, x^{\mathrm{a}}/Q']$. In both cases $R \sqsupseteq P'[x/Q'^0]$. Then

$$\begin{aligned}[\![A']\!]_\xi^{D(i,j)} = [\![C'[R]]\!]_\xi^{D(i,j)} &\sqsupseteq [\![C'[P'[x/Q'^0]]]\!]_\xi^{D(i,j)} \\ &\sqsupseteq [\![C'[(\lambda x.P')^0 Q']]\!]_\xi^{D(i,j)} \qquad \text{(by Lemma 6)} = [\![A]\!]_\xi^{D(i,j)}. \quad ∎\end{aligned}$$

THEOREM 1. *$M \in \Lambda$. $\mathcal{A}(M)$ is a directed set.*

Proof. From Lemma 7, by the Church–Rosser Theorem. ∎

As a first application of the approximation theorem, we give an interesting property about the interpretation of unsolvable terms in $D_{(i,j)}$ models. Dealing with extensional models, we first need to extend the definition of the order of a term [1, 6] to an extensional context.

[2] $C[\ ]$ denotes a context (i.e., a term with a "hole"). $C[M]$ denotes the context $C[\ ]$ where the hole has been filled with the term M.

As usual, $=_\beta$ will denote the reflexive and transitive closure of $\underset{\beta}{\rightarrow}$.

DEFINITION 6. (i) $M \in \Lambda$ is of *β-order* 0 ($\mathcal{O}(M)=0$) iff $\nexists P \in \Lambda$ such that $M =_\beta \lambda x.P$.

(ii) The *β-order* of a term M is either the integer $\mathcal{O}(M)$ such that

$$\mathcal{O}(M) = \operatorname*{Max}_n\, (M =_\beta \lambda x_1 \ldots x_n.P \wedge \mathcal{O}(P)=0)$$

or ∞ if the maximum does not exist).

(iii) The *$\beta-\eta$-order* of a term M is either the integer $O(M)$ such that

$$O(M) = \min_n\, (M =_{\beta\eta} P \wedge \mathcal{O}(P)=n)$$

or ∞ if the minimum does not exist).

PROPERTY 1. (i) *$M \in \Lambda$, M unsolvable and $O(M)=0 \Rightarrow [\![M]\!]_\xi^{D(i,j)} \in D^0_{(i,j)}$.*

(ii) *If D^0 is a lattice, $M \in \Lambda$, M unsolvable, and $O(M)=n \Rightarrow [\![M]\!]_\xi^{D(i,j)} \sqsubseteq \top_n$, where $\top_n$ is the top element of $D^n_{(i,j)}$.*

Proof. (i) Since M is unsolvable and $O(M)=0$, if $M \underset{\beta}{\twoheadrightarrow} M'$, $M' =_{\beta\eta} (\lambda x.P)QM_1 \ldots M_m$. Then $[\![M]\!]_\xi^{D(i,j)} = [\![M']\!]_\xi^{D(i,j)} = \bigsqcup \{[\![A]\!]_\xi \mid A \in \mathcal{A}(M')\}$. But $A \in \mathcal{A}(M')$ implies that A is of the shape $\Phi BCA_1 \ldots A_m$.

(ii) M unsolvable and $O(M)=n \Rightarrow M =_{\beta\eta} M'$, $M' \equiv \lambda x_1 \ldots x_n.P$ and $O(P)=0$:

$$\begin{aligned}[\![M]\!]_\xi^{D(i,j)} &= [\![M']\!]_\xi^{D(i,j)} = \bigsqcup \{[\![\lambda x_1 \ldots x_n.A]\!]_\xi \mid A \in \mathcal{A}(P)\} \\ &\sqsubseteq [\![\lambda x_1 \ldots x_n.\top_0]\!]_\xi^{D(i,j)} = \top_n \qquad (\text{since } \top_n = \lambda d : D^{n-1}_{(i,j)}.\top_{n-1}). \quad \blacksquare\end{aligned}$$

Digression on ω-Algebraic Lattices

A remarkable full subcategory of CPOs is the category of ω-algebraic lattices and Scott continuous functions. D is an ω-algebraic lattice provided it is a lattice, with a countable number of compact elements $D^c_{i,j}$, such that for all $x \in D$, $x = \bigsqcup \{y \sqsubseteq x \mid y \text{ compact}\}$, where the set on the l.h.s. is directed. Scott domains most frequently used in Computer Science are consistently complete ω-algebraic CPOs. These can be turned into ω-algebraic lattices by adding a top element.

It is an open problem whether a lambda theory induced by a CPO model is already induced by an ω-algebraic lattice model. Any solution to this problem will have to clarify the syntactic role of the "over-defined" element $\top$. All λ-models analyzed in the following sections are reflexive objects in the category of ω-algebraic lattices.

In [3] the analogy between filter λ-models and topological λ-models which are ω-algebraic lattices was first noted (for details concerning type assignment systems and filter λ-models see [2, 3]). This analogy lies in the fact that type symbols in filter λ-models play the role of names for compact elements in the corresponding topological model. In view of our further applications we generalize the isomorphism established in [3] between certain D_∞ spaces and suitable filter λ-models to all $D_{(i,j)}$ models which are ω-algebraic lattices. By means of this isomorphism we give, for all these models, another equivalent version of the Approximation Theorem.

Given any ω-algebraic lattice $D_{(i,j)}$, consider the set of type variables $V_c = \{\varphi_c \mid c$ is a compact element of D^0 different from $\perp\}$, and take the type theory T generated by the axiom schemes in [2, Sect. 2.3] and the following sets of axioms:

$$\{\varphi_{c_1} \leqslant \varphi_{c_2} \mid c_1 \geqslant c_2\}$$

and

$$\left\{\varphi_{c_{i_0}} \sim \cap (i_1, i_2) \in J \varphi_{c_{i_1}} \to \varphi_{c_{i_2}} \,\middle|\, i(c_{i_0}) = \bigsqcup (i_1, i_2) \in J f_{c_{i_1}, c_{i_2}}\right\}.$$

$D_{(i,j)}$ is isomorphic to the filter structure induced by T both as a complete lattice and as applicative structure, and there is a bijective function h: Type symbols $\to D^c_{(i,j)}$ satisfying

$$h(\omega) = \perp^3$$
$$h(\varphi_c) = c$$
$$h(\alpha \to \beta) = f_{h(\alpha), h(\beta)}$$
$$h(\alpha \cap \beta) = h(\alpha) \cap h(\beta).$$

We can define a type assignment system $\mathscr{C}$ extending the $\leqslant$ relation between types defined in [2] as follows. Fix an onto mapping f from type variables onto V_c; f extends uniquely to a mapping f^*: Type symbols $\to$ Type symbols

$$f^*(\omega) = \omega$$
$$f^*(\alpha \to \beta) = f^*(\alpha) \to f^*(\beta)$$
$$f^*(\alpha \cap \beta) = f^*(\alpha) \cap f^*(\beta)$$

and define

$$\alpha \leqslant_T \beta \Leftrightarrow f^*(\alpha) \leqslant f^*(\beta).$$

The type assignment system $\mathscr{C}$ is then defined substituting into the system in [2] the rule

$$(\leqslant)\,\frac{\sigma \leqslant \tau \;\; \sigma M}{\tau M}$$

[3] ω is the "universal"-type.

with

$$(\leqslant_T)\,\frac{\sigma \leqslant_T \tau \;\sigma M}{\tau M}.$$

The interpretation of M in $D_{(i,j)}$ can be defined as

$$[\![M]\!]^{D(i,j)} = \bigsqcup \{h(f^*(\alpha)) \mid B_\xi \overset{\mathscr{C}}{\vdash} \alpha M\},$$

where

— $B_\xi \overset{\mathscr{C}}{\vdash} \sigma M$ denotes that, from the basis B_ξ, a type σ can be deduced from M using the type assignment system $\mathscr{C}$

— B_ξ is the basis such that

$$\forall x,\ \rho x \in B \Leftrightarrow \xi(x) \supseteq \uparrow\{\rho\} \qquad \text{(or equivalently } \rho \in \xi(x)),$$

where $\uparrow\{\rho\}$ is the principal filter generated by ρ.

The type assignment system $\mathscr{C}$ can be extended to Λ^*, introducing the new rule

$$(\Phi)\,\frac{\sigma \sim \psi \to \psi}{\sigma \Phi},$$

where ψ is a type variable. Then the Approximation Theorem can be given also in the following way:

APPROXIMATION THEOREM (II). $M \in \Lambda$. $B \overset{\mathscr{C}}{\vdash} \sigma M \Leftrightarrow \exists A \in \mathcal{A}(M),\ B \overset{\mathscr{C}}{\vdash} \sigma A$.

Proof. Arguing as in [3, Theorem 4.7],

$$\begin{aligned}
(\Rightarrow)\quad & B \overset{\mathscr{C}}{\vdash} \sigma M \Leftrightarrow \uparrow\{\sigma\} \sqsubseteq [\![M]\!]_\xi^{D(i,j)} \Rightarrow \exists A \in \mathcal{A}(M),\\
& \quad \uparrow\{\sigma\} \sqsubseteq [\![A]\!]_\xi^{D(i,j)} \Rightarrow B \overset{\mathscr{C}}{\vdash} \sigma A,\\
(\Leftarrow)\quad & B \overset{\mathscr{C}}{\vdash} \sigma A \Leftrightarrow \uparrow\{\sigma\} \sqsubseteq [\![A]\!]_\xi^{D(i,j)} \Rightarrow \uparrow\{\sigma\} \sqsubseteq [\![M]\!]_\xi^{D(i,j)} \Rightarrow B \overset{\mathscr{C}}{\vdash} \sigma M,
\end{aligned}$$

where ξ is such that $\xi(x) = \uparrow\{\sigma\}$ if $\sigma x \in B$, $\uparrow\{\omega\}$ otherwise. ∎

3. PARK'S MODEL

Park's model, which we denote with $\mathscr{P}$, was first defined in [8].

In this section we use the Approximation Theorem to derive some properties of $\mathscr{P}$ and its theory.

DEFINITION 7. (i) $\mathscr{P} = D_{(i_p,\, j_p)}$, where

$$D^0 = \begin{cases} \Phi \\ \perp \end{cases}$$

and $i_p \colon D^0 \to [D^0 \to D^0]$ is such that

$$i_p(\Phi) = f_{\Phi, \Phi}$$
$$i_p(\perp) = C_\perp$$

(we use the symbol Φ instead of $\top_0$).

(ii) $\mathscr{P}$ is the filter λ-model induced by the type assignment system built on the set of variables $V = \{\varphi\}$, extending the relation $\leqslant$ between types defined in [2] with the equivalence relation $\varphi \sim \varphi \to \varphi$.

Then the results of Section 2 can be extended to $\mathscr{P}$, using the following:

LEMMA 8. $[\![\Phi]\!]^{\mathscr{P}}_{\xi} = \Phi = \uparrow\{\varphi\}$.

Proof. Trivial, by definition of the interpretation of the constant Φ given in Section 2. ∎

Let us denote with Λ^0 (Λ^{*0}) the set of terms M such that $\exists M'$, $M \twoheadrightarrow_{\beta} M'$ and $FV(M') = \varnothing$ (i.e., in M' there are not occurrences of free variables). We shall refer freely to a term of Λ^0 (Λ^{*0}) as a closed term.

PROPERTY 2 (D. Park). $M \in \Lambda^{*0}. [\![M]\!]^{\mathscr{P}} \sqsupseteq \Phi$.

Proof. It is easy to verify, by induction on the structure of M, that, starting from premises φx for every variable x occurring free in M, a deduction $D : \vdash \varphi M$ can be built. ∎

COROLLARY 2. *Let (M, I) be an indexed term, and let $I(M) = 0$. If $M \in \Lambda^{*0}$, $[\![M^I]\!]^{\mathscr{P}} = \Phi$.*

Now we define an equivalence relation between terms of Λ^*, and we prove that equivalent terms have the same interpretation in $\mathscr{P}$. Note that the converse does not hold; two terms can have the same interpretation and yet not be equivalent (see, for example, the two terms $X = \lambda x.\Phi(x\Phi)(x\lambda x.\Phi)$ and $Z = \lambda x.\Phi(x\Phi)$).

DEFINITION 8. Let $\approx$ be the equivalence relation between terms of Λ^*, induced by the equality rules

$$\{\beta, \eta, \Phi_1, \Phi_2, \Phi_3, \Phi_4\},$$

where

$$(\Phi_1)\quad \Phi M_1 M_2 \dots M_m \sim_{\Phi_1} \Phi M_2 \dots M_m \qquad \text{if} \quad M_1 \in \Lambda^{*0}$$

$$(\Phi_2)\quad \Phi M_1 \dots M_m \sim_{\Phi_2} \Phi N_1 \dots N_q \qquad \text{if} \quad \forall i\,(1 \leqslant i \leqslant m)\, \exists j\,(1 \leqslant j \leqslant q)$$
$$\wedge\ \forall j\,(1 \leqslant j \leqslant q)\, \exists i (1 \leqslant i \leqslant m) \text{ s.t. } M_i \approx N_j$$

$$(\Phi_3)\quad \Phi(\lambda x . M \sim_{\Phi_3} \Phi M[x/\Phi]$$

$$(\Phi_4)\quad \Phi(\Phi M_1 \dots M_m) \sim_{\Phi_4} \Phi M_1 \dots M_m.$$

Note that, by means of $\approx$, every nf in Λ^* is equivalent to an approximant of a λ-term.

PROPOSITION 1. *Λ^*-terms equivalent under $\approx$ are equal in $\mathscr{P}$.*

Proof. It suffices to prove that (Φ_i) equalities $(1 \leqslant i \leqslant 4)$ preserve the value of a term in $\mathscr{P}$.

(Φ_1) preserves the value of a term by Corollary 2.

(Φ_2) preserves the value in $\mathscr{P}$ since

$$[\![\Phi M_1 \dots M_n]\!]^{\mathscr{P}}_{\xi} = [\![(M_1)^0 \dots (M_n)^0]\!]^{\mathscr{P}}_{\xi} = \begin{cases} \Phi & \text{if} \quad \forall i [\![(M_i)^0]\!]^{\mathscr{P}}_{\xi} = \Phi \\ \perp & \text{otherwise.} \end{cases}$$

(Φ_3) preserves the value since

$$\begin{aligned}[\![\Phi(\lambda x . M)]\!]^{\mathscr{P}}_{\xi} &= [\![(\lambda x . M)^0]\!]^{\mathscr{P}}_{\xi} = [\![(\lambda x . M)^1\ \Phi]\!]^{P}_{\xi} \\ &= [\![(M[x/\Phi])^0]\!]^{\mathscr{P}}_{\xi} = [\![\Phi M[x/\Phi]]\!]^{\mathscr{P}}_{\xi}.\end{aligned}$$

(Φ_4) preserves the value since

$$[\![\Phi(\Phi M_1 \dots M_m)]\!]^{\mathscr{P}}_{\xi} = [\![(\Phi M_1 \dots M_m)^0]\!]^{\mathscr{P}}_{\xi} = [\![\Phi M_1^0 \dots M_m^0]\!]^{\mathscr{P}}_{\xi} = [\![\Phi M_1 \dots M_m]\!]^{\mathscr{P}}_{\xi}. \quad ∎$$

Remember that the ω-rule is defined [1] by

$(\omega)\quad \forall Z \in \Lambda^0\ MZ = NZ \Rightarrow M = N.$

As an aside we note that:

THEOREM 2. *The ω-rule fails in $\mathscr{P}$.*

Proof. Consider the two terms $\Delta\Delta$ and $\lambda x . \Delta\Delta$, where $\Delta = \lambda x . xx$. By Property 2, $M \in \Lambda^0$ implies $[\![M]\!]^{\mathscr{P}} \sqsupseteq \Phi$. Then $[\![\Delta\Delta M]\!]^{\mathscr{P}} = [\![(\lambda x . \Delta\Delta) M]\!]^{\mathscr{P}} = \Phi$, by Property 1. Moreover $[\![\Delta\Delta]\!]^{\mathscr{P}} = \Phi$ and $[\![\lambda x . \Delta\Delta]\!]^{\mathscr{P}} = \top_1 \sqsupsetneq \Phi$. ∎

We consider the failure of the ω-rule in $\mathscr{P}$ again in Section 5.

We now study the behaviour of fixed point operators in $\mathscr{P}$.

Park [8] proved that the interpretation in $\mathscr{P}$ of the term $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ (which is a fixed point combinator in the theory of λ-β-conversion ($\Lambda\beta$) is such that:

> Yx produces the minimal fixpoint of x which contains Φ if $[\![x]\!]_\xi^{\mathscr{P}} \sqsupseteq \Phi$, and the "correct" minimal fixpoint otherwise.

We call "fixed point operator" (in $\Lambda\beta$) a term Ξ with the property $\Xi f =_\beta f_i(\Xi f)$; if Ξ is a closed term we call it a "fixed point combinator." We prove that all fixed point combinators have the same interpretation in $\mathscr{P}$.

First we need a technical lemma.

In the sequel, we use the notation $x^{(n)}y$ to denote the term $\underbrace{x(x\ldots(xy)\ldots)}_{n}$.

Lemma 9. *Let Ξ be a fixed point operator in $\Lambda\beta$. Then for each $p \geqslant 0$, there are a term R and an integer $n \geqslant 0$ such that*

(i) *R is not in head normal form*

(ii) *$\Xi f \xrightarrow[\beta]{(m)}\!\!\!\!\twoheadrightarrow f^{(n)}R \twoheadrightarrow_\beta f^{(n+1)}R'$, where R' is not a head normal form, and the reduction from Ξf to $f^{(n)}R$ is done in $m \geqslant p$ steps.*

Proof. By induction on p.

$p = 0$. Since Ξ is a fixed point operator, $\Xi f =_\beta f(\Xi f)$; then, by the Church–Rosser theorem, there exists a term P such that

$$\Xi f \twoheadrightarrow_\beta P \twoheadleftarrow_\beta f(\Xi f),$$

and P must be of the shape fP'.

Let $P = f^{(n+1)}P'$, where P' is not an hnf. By the standardization theorem [1], there are two standard reductions σ and σ':

$$\sigma\colon \Xi f \twoheadrightarrow_\beta f^{(n+1)}P'$$
$$\sigma'\colon \Xi f \twoheadrightarrow_\beta f^{(n)}P'.$$

Let σ_0 be the leftmost outermost subreduction included in σ'. Then $\sigma = \sigma_0 + \sigma_1$, where $\sigma_0\colon \Xi f \twoheadrightarrow_\beta f^{(n)}R$ (since $f^{(n)}R$ starts with n variables), $\sigma_1\colon f^{(n)}R \twoheadrightarrow_\beta f^{(n)}P'$ (σ_1 can be empty), and R is not in hnf since P' is not in hnf.

Since $f^{(n+1)}P'$ starts with $n+1$ variables, the leftmost outermost part of σ must include σ_0, so we can split σ as $\sigma_0 + \sigma_{2'}$ where $\sigma_2\colon f^{(n)}R \twoheadrightarrow_\beta f^{(n+1)}P'$.

The induction step is similar since $\Xi f \twoheadrightarrow_\beta f^{(n)}R$ implies $R = \Xi f$, and so R has the same properties of Ξf. ∎

Theorem 3. *Let Ξ be a fixed point combinator:*

$$[\![\Xi]\!]^{\mathscr{P}} = \bigsqcup_n \{[\![\lambda f\cdot f^{(n)}(\Phi f)]\!] \mid n \geqslant 0\}.$$

Proof. Let Ξ be a fixed point combinator in $\Lambda\beta$. By the approximation theorem $[\![\Xi f]\!]_\xi^{\mathscr{P}} \sqsupseteq \bigsqcup_n \{[\![f^{(n)}(\Phi f)]\!]_\xi^{\mathscr{P}} \mid n \geqslant 0\}$, Ξ being a closed term.

In order to prove the converse, we show that, for each indexing I, $[\![(\Xi f, I)]\!]_\xi^{\mathscr{P}} \sqsubseteq [\![f^{n_1}(\ldots(f^{n_i}(\Phi f^0))\ldots)]\!]_\xi^{\mathscr{P}}$. To do this, we define a new notion of 0-indexed reduction:

$$\text{(R)} \quad (\lambda x.P^s)^0\, Q^r \xrightarrow[R]{} (P[x/Q^0])^0.$$

This reduction can possibly increase the value of a term, since $[\![(\lambda x.P^s)^0 Q^0]\!]_\xi^{\mathscr{P}} \sqsubseteq [\![(P[x/Q^0])^0]\!]_\xi^{\mathscr{P}}$.

Given $(\Xi f, I)$ let (P, J) be its wnf (see Lemma 3).

Applying the leftmost outermost (R)-reduction strategy to (P, J), after a suitable number of steps, by Lemma 9, we operate a reduction step of the shape

$$(f^{n_1}\ldots(f^{n_i}(R)^k)^{m_i}\ldots)^{m_1} \xrightarrow[R]{}\!\!\!\!\rightarrow (f^{n_1}\ldots f^{n_i}(f^r(R')^h)^{m_{i+1}})^{m_i}\ldots)^{m_1},$$

where both R and R' are indexed terms not in hnf, whose head redexes have index 0.

Then $[\![R]\!]_\xi^{\mathscr{P}} = [\![R^0]\!]_\xi^{\mathscr{P}}$ and $[\![R']\!]_\xi^{\mathscr{P}} = [\![R'^0]\!]_\xi^{\mathscr{P}}$ (by Lemma 6). Then

$$(R)^0 \xrightarrow[R]{}\!\!\!\!\rightarrow (f^r(R'))^0$$

and

$$[\![(f^r(R')^0)^0]\!]_\xi^{\mathscr{P}} \sqsubseteq [\![(f^r\Phi)^0]\!]_\xi^{\mathscr{P}} = [\![f^0]\!]_\xi^{\mathscr{P}} = [\![\Phi f]\!]_\xi^{\mathscr{P}}.$$

Hence

$$[\![\Xi f]\!]_\xi^{\mathscr{P}} = \bigsqcup_n \{[\![f^{(n)}(\Phi f)]\!]_\xi^{\mathscr{P}} \mid n \geqslant 0\}. \quad \blacksquare$$

Remark 1. It is interesting to contrast Park's model theory and $\mathscr{H}^*$ [1] with respect to fixed point operators.

Let Y_Z be the fixed point operator $\lambda f.(\lambda xy.f(xxy))(\lambda xy.f(xxy))z$. Y_z and Y have the same Böhm tree, hence $\mathscr{H}^* \vdash Y_z = Y$, but $[\![Y_z]\!]_\xi^{\mathscr{P}} \subsetneq [\![Y]\!]_\xi^{\mathscr{P}}$, for suitable assignments ξ ($\xi(z) = \bot$ for instance) since

$$\begin{aligned} \mathscr{A}(Y_z) &= \{\lambda f.f^{(n)}(\Phi(\lambda xy.f(xxy))(\lambda xy.f(xxy))z) \mid n \geqslant 0\} \\ &= \{\lambda f.f^{(n)}(\Phi fz)\} \qquad \text{(from the equivalence relation } \approx \text{).} \end{aligned}$$

Therefore the hypothesis that Ξ is a combinator is essential in Theorem 3. This is a first example of the different behaviour of open and closed terms in $\mathscr{P}$.

Remark 2. The ω-rule in Park's model fails also for head normal forms, and hence $\mathscr{H}^*$ is a coarser equivalence relation than $\mathscr{P}$ also when restricted to closed hnfs.

Put $Y^2 = \lambda f.(\lambda x.f(f(xx)))(\lambda x.f(f(xx)))$, i.e., $Y^2 f = Y(f \circ f)$. Y^2 and Y have the same Böhm tree, hence $\mathscr{H}^* \vdash Y^2 = Y$. In the interior of Park's model Y^2 and Y coincide. But $[\![Y^2]\!]^{\mathscr{P}} \neq [\![Y]\!]^{\mathscr{P}}$ since

$$\Phi = [\![Y^2]\!]^{\mathscr{P}} (f_{\Phi, f_{\lambda x.\Phi, \lambda x.\Phi}} \sqcup f_{f_{\lambda x.\Phi, \lambda x.\Phi}, \Phi}) \neq [\![Y]\!]^{\mathscr{P}} (f_{\Phi, f_{\lambda x.\Phi, \lambda x.\Phi}} \sqcup f_{f_{\lambda x.\Phi, \lambda x.\Phi}, \Phi}) = \bot.$$

We now show that the theory of $\mathscr{P}$ is not semisensible; in fact, every solvable term is equated to an unsolvable one. By the very same argument we also show that every normal form is equated in $\mathscr{P}$ to a head normal form.

Let $B = \lambda xyz.x(yz)$; it is easy to see that YB is an unsolvable term, and $O(YB) = \infty$. Following [12], let $F = \lambda xyz.y(xz)$ and let $J = YF$ (recall that J is an hnf with an infinite Böhm tree).

THEOREM 4. $[\![I]\!]^{\mathscr{P}} = [\![J]\!]^{\mathscr{P}} = [\![YB]\!]^{\mathscr{P}}$.

Proof. In this proof we use the notation X_n to indicate $([\![X]\!]^{\mathscr{P}})_n$. Let M be either B or F. $[\![YM]\!]^{\mathscr{P}} \sqsubseteq [\![I]\!]^{\mathscr{P}}$ by Theorem 3, since $MI =_\beta I$.

For the converse, we prove inductively that $\forall n\ [\![M^{(n)}\Phi]\!]^{\mathscr{P}} \sqsupseteq I_n$:

$n = 1, \qquad [\![M^{(1)}\Phi]\!]^{\mathscr{P}} \sqsupseteq \Phi$ since M is a closed term (see Property 2);

$$n = k+1 \qquad M \equiv B \qquad [\![(B^{(k+1)}\Phi)xy]\!]^{\mathscr{P}}_\xi = [\![(B^{(k)}\Phi)(xy)]\!]^{\mathscr{P}}_\xi \sqsupseteq I_k \cdot [\![(xy)]\!]^{\mathscr{P}}_\xi$$

(by induction hypothesis) $= (xy)_{k-1}$

$$M \equiv F \qquad [\![(F^{(k+1)}\Phi)xy]\!]^{\mathscr{P}}_\xi = [\![x(F^{(k)}\Phi y)]\!]^{\mathscr{P}}_\xi \sqsupseteq [\![x]\!]^{\mathscr{P}}_\xi \cdot (I_k \cdot [\![y]\!]^{\mathscr{P}}_\xi)$$

(by induction hypothesis) $= [\![x]\!]^{\mathscr{P}}_\xi \cdot y_{k-1}$.

Both $(xy)_{k-1}$ and $[\![x]\!]^{\mathscr{P}}_\xi \cdot y_{k-1}$ are greater than

$$x_k \cdot [\![y]\!]^{\mathscr{P}}_\xi = (x_k\, y_{k-1})_{k-1}.$$

The thesis follows, since $[\![YM]\!]^{\mathscr{P}} = \bigsqcup_n \{[\![M^{(n)}\Phi]\!] \mid n \geq 0\}$, by Theorem 3. ∎

4. THE MODEL $\mathscr{T}$

In this section we define a model, which we call $\mathscr{T}$, and we apply to it the technique of Section 2 to study some of its properties.

DEFINITION 9. (i) $\mathscr{T} = D_{(i_{\mathscr{T}}, j_{\mathscr{T}})}$, where

$$D^0 = \begin{cases} \top_0 \\ \Phi \\ \bot_0 \end{cases}$$

and $i_{\mathscr{T}}: D^0 \to [D^0 \to D^0]$ is such that

$$i_{\mathscr{T}}(\bot_0) = C_{\bot_0} = f_{\bot_0, \bot_0}$$
$$i_{\mathscr{T}}(\Phi) = f_{\Phi,\Phi} \sqcup f_{\top_0,\top_0}$$
$$i_{\mathscr{T}}(\top_0) = f_{\Phi,\top_0}.$$

(ii) $\mathscr{T}$ is the filter λ-model induced by the type assignment built on the set of variables $V = \{\varphi, \psi\}$, extending the relation $\leqslant$ between types defined in [2] with the following equivalence relations:

$$\varphi \sim \varphi \to \varphi \cap \psi \to \psi$$
$$\psi \sim \varphi \to \psi$$
$$\psi \leqslant \varphi.$$

Then the results of Section 2 can be extended to $\mathscr{T}$, using the following:

LEMMA 10. $[\![\Phi]\!]_{\xi}^{\mathscr{T}} = \Phi = \uparrow\{\varphi\}$.

The proof is obvious.

DEFINITION 10. (i) [1] The set of terms $\bar{\Lambda}_I$ ($\lambda - I$-calculus) is defined as follows:

$$x \in \bar{\Lambda}_I$$
$$M \in \bar{\Lambda}_I, \quad x \in FV(M) \Rightarrow (\lambda x . M) \in \bar{\Lambda}_I$$
$$M, N \in \bar{\Lambda}_I \Rightarrow (MN) \in \bar{\Lambda}_I.$$

(ii) The class of terms $\bar{\Lambda}_I^*$ is obtained by adjoining to (i) the clause $\Phi \in \bar{\Lambda}_I$.

(iii) Let $\Lambda_I(\Lambda_I^*)$ be the set of terms M such that $\exists M'$, $M \xrightarrow[\beta]{} \!\!\!\!\rightarrow M'$ and $M' \in \bar{\Lambda}_I$ $(\bar{\Lambda}_I^*)$. Let us denote by $\Lambda_I^0(\Lambda_I^{*0})$ the set of closed terms of $\Lambda_I(\Lambda_I^*)$.

PROPERTY 3. $M \in \Lambda_I^{*0}. [\![M]\!]^{\mathscr{T}} \sqsupseteq \Phi$.

Proof. Recall that the combinatory version of $\bar{\Lambda}_I$ has, as generators, the combinators I, B, C, and S. Then it is sufficient to show that each one of these has an interpretation in $\mathscr{T}$ equal to or greater than Φ.

It can be easily checked that any combination of Φ and $\top_0$ has a value of either Φ or $\top_0$, the latter case occurring iff there is at least one occurrence of $\top_0$. Let $F = \lambda x_1 \dots x_n . A$, where A is a pure combination (i.e., a term without abstractions) and for all i $(1 < i < n)$ $x_i \in FV(A)$. Then $[\![F]\!]^{\mathscr{T}} \sqsupseteq \Phi$ since $[\![F]\!]^{\mathscr{T}} \zeta_1 \dots \zeta_n$ (where $\zeta_i \in \{\Phi, \top_0\}$ $(1 \leqslant 1 \leqslant n)$) is a pure combination build on $\{\zeta_1, ..., \zeta_n\}$.

Then the proof follows, since

$$I \equiv \lambda x.x$$

$$B \equiv \lambda xyz.x(yz)$$

$$S \equiv \lambda xyz.xz(yz)$$

$$C \equiv \lambda xyz.xzy$$

have the same structure as F. ∎

LEMMA 11. *Let ξ be the environment such that $\forall x.\xi(x) = \Phi$. Then for all $M \in \Lambda$, $[\![M]\!]_\xi^{\mathscr{T}} \neq \top_0$.*

Proof. By the Approximation Theorem, it is sufficient to prove that there is no approximant A such that $[\![A]\!]_\xi^{\mathscr{T}} \sqsupseteq \top_0$ (since $\top_0$ is a compact element). We prove this by induction on the structure of approximants:

$A \equiv x$ obvious.

$A \equiv \Phi A_1 \ldots A_m$ follows immediately from the induction hypothesis.

$A \equiv x A_1 \ldots A_m$ is the same as the preceding case, since by hypothesis $\xi(x) = \Phi$.

$A \equiv \lambda x.A'$ by the induction hypothesis. ∎

From Property 3 and Lemma 11 we have:

COROLLARY 3. *Let (M, I) be an indexed term, and let $I(M) = 0$. If $M \in \Lambda_I^{*0}$, $[\![M^I]\!] = \Phi$.*

DEFINITION 11. Let $\approx_I$ be the equivalence relation betwen terms of Λ^*, induced by the equality rules

$$\{\beta, \eta, \Phi_1', \Phi_2, \Phi_4\},$$

where Φ_2, Φ_4 are defined in Definition 6, and

$$(\Phi_1') \quad \Phi M_1 M_2 \ldots M_m \sim_{\Phi_1'} \Phi M_2 \ldots M_m \qquad \text{if} \quad M_1 \in \Lambda_I^{*0}.$$

PROPOSITION 2. Λ^* terms equivalent under $\approx_I$ are equal in $\mathscr{T}$.

Proof. The equivalence induced by Φ_1' preserves equality of terms, by Corollary 3. The proof for the equivalence relations induced by Φ_2 and Φ_4 is similar to the proof of Proposition 1, using Corollary 3 and taking into account that, in $\mathscr{T}$,

$$[\![\Phi M_1 \ldots M_m]\!]_\xi^{\mathscr{T}} = \begin{cases} \Phi & \text{if} \quad \forall i.[\![M_i^0]\!]_\xi^{\mathscr{T}} = \Phi. \\ \top_0 & \text{if} \quad \forall i.[\![M_i^0]\!]_\xi^{\mathscr{T}} \sqsupseteq \Phi \qquad \text{and} \qquad \exists i.[\![M_i^0]\!]_\xi^{\mathscr{T}} = \top_0. \\ \bot_0 & \text{otherwise.} \end{cases}$$ ∎

In $\mathscr{T}$, as in $\mathscr{P}$, the ω-rule fails, as can be seen in the following:

THEOREM 5. The ω-rule fails in $\mathscr{T}$.

Proof. Consider the two terms $M \equiv \lambda y.\Delta\Delta(\lambda x.y(y(\Delta\Delta)))$ and $N \equiv \lambda y.\Delta\Delta(\lambda x.y)$. $[\![M]\!]^{\mathscr{T}} \neq [\![N]\!]^{\mathscr{T}}$, since

$$\begin{aligned} \top_0 = [\![M]\!]^{\mathscr{T}}\ (f_{\Phi,\lambda x.\Phi} \sqcup f_{\lambda x.\Phi,\top_0}) \\ \neq [\![N]\!]^{\mathscr{T}}\ (f_{\Phi,\lambda x.\Phi} \sqcup f_{\lambda x.\Phi,\top_0}) = \bot_0. \end{aligned}$$

Let $L \in \Lambda^0$. $[\![ML]\!]^{\mathscr{T}} = [\![NL]\!] = \bot_0$, since $[\![\Delta\Delta]\!]^{\mathscr{T}} = \Phi$ (by Property 1 and Corollary 3) and $[\![(\lambda x.L(L(\Delta\Delta)))^0]\!]^{\mathscr{T}} = [\![(\lambda x.L)^0]\!]^{\mathscr{T}} = \bot_0$ (by Lemma 14, which is proved in the next section, since both these terms do not belongs to Λ_I). ∎

Now, we give some properties of the interpretation of terms in $\mathscr{T}$.

THEOREM 6. $[\![I]\!]^{\mathscr{T}} = [\![J]\!]^{\mathscr{T}} = [\![YB]\!]^{\mathscr{T}}$.

Proof. To prove this theorem, we need the following:

FACT. *Let Ξ be a fixed point combinator belonging to Λ_I. Then, if $M \in \Lambda_I^0$, $[\![\Xi M]\!]^{\mathscr{T}} = \bigsqcup_n \{[\![M^{(n)}\Phi]\!]^{\mathscr{T}} \mid n \geqslant 0\}$.*

Proof. By Lemma 4, $[\![\Xi]\!]^{\mathscr{T}}_\xi = \bigsqcup_n \{[\![\lambda f.f^{(n)}(\Xi f)^0]\!]^{\mathscr{T}} \mid n \geqslant 0\}$. Then $[\![\Xi M]\!]^{\mathscr{T}} = \bigsqcup_n \{[\![M^{(n)}(\Xi M)^0]\!]^{\mathscr{T}} \mid n \geqslant 0\}$. But $\Xi \in \Lambda_I^0$ and $M \in \Lambda_I^0$ implies $\Xi M \in \Lambda_I^0$, and the result follows from Corollary 3. Now the proof of Theorem 4 applies to $\mathscr{T}$, since I, F, and B belong to Λ_I^0. ∎

Remark 3. It is interesting to note that not all fixed point combinators are equal in $\mathscr{T}$, differently from $\mathscr{P}$. Consider the fixed point combinator

$$Y_k \equiv \lambda f.(\lambda xy.f(xxy))(\lambda xy.f(xxy))K, \qquad \text{where} \quad K \equiv \lambda xy.x,$$

$$\begin{aligned} [\![Y_k]\!]^{\mathscr{T}} &= \bigsqcup_n \{[\![\lambda f.f^{(n)}(\Phi(\lambda xy.f(xxy))(\lambda xy.f(xxy))K)]\!]^{\mathscr{T}} \mid n \geqslant 0\} \\ &= \bigsqcup_n \{[\![\lambda f.f^{(n)}\bot_0]\!]^{\mathscr{T}} \mid n \geqslant 0\}. \end{aligned}$$

In fact $[\![K^0]\!]^{\mathscr{T}} = \bot_0$, since otherwise $[\![K^0]\!]^{\mathscr{T}} \sqsupseteq \Phi \Rightarrow [\![\lambda y.x]\!]^{\mathscr{T}}_\xi \sqsupseteq \Phi$, where $\xi(x) = \Phi$, which is impossible by Lemma 11.

5. CONTEXTUAL THEORIES

Morris [7, see also 1] proved the following theorem:

THEOREM 7. *Let $Q \subseteq \Lambda$ be non-trivial ($\neq \phi$, $\neq \Lambda$) and closed under α- and β-equality. Define*

(i) $M \sim_Q N \Leftrightarrow (\forall C[\ ], C[M] \in Q \Leftrightarrow C[N] \in Q)$

(ii) $\mathscr{C}_Q = \{M = N \mid M, N \in \Lambda^0 \text{ and } M \sim_Q N\}$.

Then

$$\sim_Q \quad \text{is a congruence relation}$$

$$\mathscr{C}_Q \quad \text{is a } \lambda\text{-theory.}$$

We call *contextual* every λ-theory which arises from a set of terms Q closed under α- and β-equality, by Theorem 7. We call Q the set of *observables* of the theory.

Scott [11] proves the following:

THEOREM 8. *Every λ-theory is contextual.*

Proof. Given a λ-theory $\mathscr{C}$, put

$$Q = \{M \mid \exists A, B\ M =_{\mathscr{C}} \langle A, B\rangle \wedge A =_{\mathscr{C}} B\},$$

where $\langle A, B\rangle \equiv \lambda x.xAB$, and x does not occur in A or B. Trivially $\mathscr{C} \subseteq \mathscr{C}_Q$. Conversely, if $M \neq_{\mathscr{C}} N$, let $C[\ ]$ be the context $\lambda x.x[\ ]N$. Obviously $C[N] \in Q$. If $C[M] \in Q$, we would have a contradiction, since $C[M] =_{\mathscr{C}} \lambda x.xAB$, where $A =_{\mathscr{C}} B$. But then, applying both sides once to K and once to $0 = \lambda xy.y$, we obtain respectively

$$M =_{\mathscr{C}} A$$

$$N =_{\mathscr{C}} B$$

which implies $M =_{\mathscr{C}} N$. ∎

Note that different sets of observables can give rise to the same λ-theory. However, given a λ-theory $\mathscr{C}$, it is useful to look for a particular set of observables which gives the best insight into the operational behaviour of $\mathscr{C}$.

Let $\mathscr{C}^{\mathscr{P}}$ be the theory of Park's model. The most relevant operational properties of $\mathscr{C}^{\mathscr{P}}$ are

— the failure of the ω-rule

— the equivalences Φ_1 between approximants.

Both these properties suggest that $\mathscr{C}^{\mathscr{P}}$ is sensitive to free occurrences of a variable in a term. We therefore compare $\mathscr{C}^{\mathscr{P}}$ with the contextual theory obtained by choosing as Q the set of closed terms. It turns out the theory of Park's model is strictly included in $\mathscr{C}_{\Lambda^0}$ (Theorems 9, 10, and 11).

THEOREM 9.

$$\mathscr{C}^{\mathscr{P}} \subseteq \mathscr{C}_{\Lambda^0}.$$

In order to prove this theorem, we need the following:

LEMMA 12. *Let $A \in \Lambda^*$ be an nf, and let $FV(A) = \varnothing$. Then for every environment ξ such that $\exists x \in FV(A)$, $\xi(x) = \bot$, and $\forall y \neq x\ \xi(y) = \Phi$, $[\![A]\!]^{\mathscr{P}}_{\xi} \not\sqsupseteq \Phi$.*

Proof. By structural induction.

— $A \equiv x$. Obvious.

— $A \equiv \lambda x.A'$. Since $FV(A) \neq \varnothing$, there is at least one variable $y \not\equiv x$ which occurs free in A'. Let ξ be an environment satisfying the condition with respect to A.

By the induction hypothesis, $[\![A']\!]^{\mathscr{P}}_{\xi[x/\Phi]} \not\sqsupseteq \Phi$ and, since $[\![A]\!]^{\mathscr{P}}_{\xi}\Phi = [\![A']\!]^{\mathscr{P}}_{\xi[x/\Phi]}$, $[\![A]\!]^{\mathscr{P}}_{\xi} \not\sqsupseteq \Phi$.

— $A \equiv \Phi A_1 \dots A_m$. For every ξ satisfying the condition w.r.t. A, $\exists j. [\![A_j]\!]^{\mathscr{P}}_{\xi} \not\sqsupseteq \Phi$. Hence the thesis follows by the induction hypothesis.

— $A \equiv xA_1 \dots A_m$. Every suitable ξ is such that $\xi(x)$ can be either $\bot$ or Φ. The first case is obvious, the second is similar to the preceding one. ∎

Proof of Theorem 9. We actually prove a statement equivalent to the theorem, namely: No open term (i.e., a term which does not belong to Λ^0) can be equated to a closed term in Park's theory.

First of all, we state the following property:

PROPERTY 4. *If M is an open term, then there is z such that $\forall A \in \mathscr{A}(M)$, $z \in FV(A)$.*

(By the Church–Rosser Theorem, and by the Approximation Theorem.)

Now, let M be an open term. Then, by Lemma 12 and by Property 4, there is ξ such that $\forall A \in \mathscr{A}(M)$, $[\![A]\!]^{\mathscr{P}}_{\xi} \not\sqsupseteq \Phi$, then $[\![M]\!]^{P}_{\xi} \not\sqsupseteq \Phi$.

Since, by Property 2, $N \in \Lambda^0 \Rightarrow [\![N]\!] \sqsupseteq \Phi$, the theorem is proved. ∎

In order to prove that $\mathscr{C}^{\mathscr{P}}$ is strictly included in $\mathscr{C}_{\Lambda^0}$ we have to exhibit two terms, say X and Z, such that $\mathscr{C}^{\mathscr{P}} \not\vdash X = Z$ while $\mathscr{C}_{\Lambda^0} \vdash X = Z$. Define

$$X \equiv \lambda xz.\Delta\Delta(x(\Delta\Delta z)(\Delta\Delta))(x(\Delta\Delta)(\Delta\Delta z))$$

$$Z \equiv \lambda xz.\Delta\Delta(x(\Delta\Delta z)(\Delta\Delta z)).$$

Since X and Z are closed terms, proving $\mathscr{C}_{\Lambda^0} \vdash X = Z$ amounts to proving that for every term P, PX reduces to a closed term iff PZ reduces to a closed term. We can restrict P to range only over approximants. In fact, consider the environment $\xi: \mathrm{Var} \to \mathscr{P}$ such that $\forall x \xi(x) = \bot$. PX is a closed term iff $[\![PX]\!]^{\mathscr{P}}_{\xi} \sqsupseteq \Phi$ iff $\exists P' \in \mathscr{A}(P)[\![P'X]\!]^{\mathscr{P}}_{\xi} \sqsupseteq \Phi$ iff $P'X$ is a closed term. Moreover, since $\forall \xi [\![\Delta\Delta]\!]^{\mathscr{P}}_{\xi} = \Phi$, we consider in the following, in place of X and Z, their direct approximants X' and Z', namely $X' = \lambda xz.\Phi(x(\Phi z)\Phi)(x\Phi(\Phi z))$ and $Z' = \lambda xz.\Phi(x(\Phi z))(\Phi z))$. It is interesting to note again what useful tools the notion of approximant and the approximation theorem are in the study of λ-theories.

First we need a definition and a lemma.

DEFINITION 12. (i) $M \in \Lambda$. $x \in FV^*(M)$ iff $\forall M' M \twoheadrightarrow_\beta M'$, $x \in FV(M')$.

(ii) $M \in \mathcal{A}$. M is a K-eraser iff $x_k \notin FV^*(Mx_1x_2 \dots x_k \dots x_m)$ $(m \geqslant k)$, where $x_1, \dots, x_m \notin FV(M)$.

Informally an approximant M is a k-eraser iff the kth abstracted variable of M does not occur in the body of M.

LEMMA 13. *Let* $M, M', N, N' \in \mathcal{A}$ *be such that* $FV^*(M) = FV^*(M')$, $FV^*(N) = FV^*(N')$ *and* $\forall k M$ *is a* k*-eraser iff* M' *is a* k*-eraser. Then* $FV^*(X'MN) = FV^*(Z'M'N')$.

Proof. Since the behaviour of a term of the shape ΦQ, where Q is a normal form, is the behaviour of a variable, $X'MN$ and $Z'M'N'$ reduce to normal form after a number of reductions less than or equal to 6, depending on the number of initial abstractions of M and M'. By case analysis, it is easy to see that, if both M and M' are 1 and 2-erasers, then

$$FV^*(X'MN) = FV^*(Z'M'N') = FV^*(M) = FV^*(M'), \qquad \text{otherwise}$$

$$FV^*(X'MN) = FV^*(Z'M'N') = FV^*(M) \cup FV^*(N) = FV^*(M') \cup FV^*(N'). \quad \blacksquare$$

We are now ready to prove:

THEOREM 10. $\mathscr{C}_{\Lambda^0} \vdash X = Z$.

Proof. We prove, by induction on the length of the approximant P, that, for every $P \in \mathcal{A}$:

(i) $FV^*(PX') = FV^*(PZ')$

(ii) $\forall k PX'$ is a k-eraser iff PZ' is a k-eraser.

For (i) note that both PX' and PZ' reduce to nf, $\forall P \in \mathcal{A}$. Let $P = \lambda x_1 \dots x_n . \zeta P_1 \dots P_m$ $(n \geqslant 1, m \geqslant 2)$, where ζ can be either Φ or a variable different from x_1, or x_1. In the first two cases, applying the induction hypothesis to $\lambda x_1 . P_i$ $(1 \leqslant i \leqslant m)$, we obtain $\forall i FV^*(P_i[x_1/X']) = FV^*(P_i[x_1/Z'])$ and the proof follows immediately.

In the case $\zeta \equiv x_1$, then

$$PX' \equiv \lambda x_2 \dots x_n . X'P_1[x_1/X'] \dots P_m[x_1/X']$$

$$PZ' \equiv \lambda x_2 \dots x_n . Z'P_1[x_1/Z'] \dots P_m[x_1/Z'].$$

By the induction hypothesis, $\forall i FV^*(P_i[x_1/X']) = FV^*(P_i[x_1/Z'])$ and $\forall k P_i[x_1/X']$ is a k-eraser iff $P_i[x_1/Z']$ is a k-eraser. From Lemma 13 we get

$$FV^*(X'P_1[x_1/X']P_2[x_1/X']) = FV^*(Z'P_1[x_1/Z']P_2[x_1/Z']).$$

Since by the nature of X' and Z'

$$FV^*(X'P_1[x_1/X']\ldots P_m[x_1/X'])$$
$$= FV^*(X'P_1[x_1/X']P_2[x_1/X']) \bigcup_{3\leqslant i\leqslant m} FV^*(P_i[x_1/X']) \quad \text{and}$$
$$FV^*(Z'P_1[x_1/Z']\ldots P_m[x_1/Z'])$$
$$= FV^*(Z'P_1[x_1/Z']P_2[x_1/Z']) \bigcup_{3\leqslant i\leqslant m} FV^*(P_i[x_1/Z']),$$

the proof follows immediately.

The remaining cases ($n=0$ or $m<2$) are similar but simpler.

The proof of (ii) is tedious but routine. ∎

It remains to prove that $\mathscr{C}^{\mathscr{P}} \not\vdash X=Z$.

We prove a stronger result, namely that there is no topological model which induces $\mathscr{C}_{\Lambda^0}$. We therefore answer negatively to question (ii), raised in the Introduction.

THEOREM 11. *Let D be a topological λ-model in the ccc of CPO's. If $\mathscr{C}^D \subseteq \mathscr{C}_{\Lambda^0}$, then $\mathscr{C}^D \subsetneq \mathscr{C}_{\Lambda^0}$.*

Proof. Let us use $[\![\ ]\!]$ to denote $[\![\ ]\!]^D$. Assume $\mathscr{C}^D = \mathscr{C}_{\Lambda_0}$. This implies:

(i) $[\![\Delta\Delta]\!]\cdot[\![\Delta\Delta]\!]=[\![\Delta\Delta]\!]$

(ii) $\forall d\in D[\![\Delta\Delta]\!]\cdot([\![\Delta\Delta]\!]\cdot d)=[\![\Delta\Delta]\!]\cdot d$

(iii) $[\![\Delta\Delta]\!]\neq[\![\lambda x.\Delta\Delta]\!]$; i.e., $[\![\Delta\Delta]\!]$ cannot be a constant function.

Two cases arise:

(1) $\forall a\in D[\![\Delta\Delta a]\!]\sqsubseteq[\![\Delta\Delta]\!]$. This implies $[\![\Delta\Delta]\!]\sqsubsetneq[\![\lambda x.\Delta\Delta]\!]$ and $[\![\Delta\Delta]\!]\cdot\bot\sqsubsetneq[\![\Delta\Delta]\!]$ (by iii). As remarked in the Introduction, the topological models under consideration are continuously complete; i.e., all continuous functions are representable. Consider the function

$$f(z)=\begin{cases}\lambda x.[\![\Delta\Delta]\!] & \text{if } z\sqsubseteq\Gamma[\![\Delta\Delta]\!]\cdot\bot\\ [\![\Delta\Delta]\!] & \text{otherwise;}\end{cases}$$

f is continuous. In fact, let $X\subseteq D$ be a direct set: $\sqcup X\not\sqsubseteq[\![\Delta\Delta]\!]\cdot\bot$ implies that there exists at least an element $x\in X$ such that $x\not\sqsubseteq[\![\Delta\Delta]\!]\cdot\bot$, and $\sqcup X\sqsubseteq[\![\Delta\Delta]\!]\cdot\bot$ implies that $\forall x\in X\, x\sqsubseteq[\![22]\!]\cdot\bot$. Consequently $f(\sqcup X)=\bigsqcup_{x\in X}f(x)$. Then

$$[\![X]\!]\cdot f\cdot\bot=[\![\Delta\Delta]\!]\cdot(f\cdot([\![\Delta\Delta]\!]\cdot\bot)\cdot[\![\Delta\Delta]\!])(f\cdot[\![\Delta\Delta]\!]\cdot([\![\Delta\Delta]\!]\cdot\bot))$$
$$=[\![\Delta\Delta]\!]\cdot([\![\Delta\Delta]\!]\cdot[\![\Delta\Delta]\!])\cdot[\![\Delta\Delta]\!]=[\![\Delta\Delta]\!]\quad\text{(by i),}$$

while

$$\begin{aligned}[\![Z]\!] \cdot f \cdot \perp &= [\![\Delta\Delta]\!] \cdot (f \cdot ([\![\Delta\Delta]\!] \cdot \perp) \cdot ([\![\Delta\Delta]\!] \cdot \perp)) \\ &= [\![\Delta\Delta]\!] \cdot [\![\Delta\Delta]\!] \cdot ([\![\Delta\Delta]\!] \cdot \perp) = [\![\Delta\Delta]\!] \cdot \perp \qquad \text{(by ii)}.\end{aligned}$$

Contradiction.

(2) $\exists a \in D\ [\![\Delta\Delta a]\!] \not\sqsubseteq [\![\Delta\Delta]\!]$. In this case $[\![\Delta\Delta]\!] \cdot \perp \sqsubsetneq [\![\Delta\Delta]\!] \cdot a$: in fact, $[\![\Delta\Delta]\!] \cdot \perp \sqsubseteq [\![\Delta\Delta]\!] \cdot a$ and $[\![\Delta\Delta]\!] \cdot \perp \sqsubseteq [\![\Delta\Delta]\!] \cdot [\![\Delta\Delta]\!] = [\![\Delta\Delta]\!]$ (by monotonicity); then $[\![\Delta\Delta]\!] \cdot \perp = [\![\Delta\Delta]\!] \cdot a$ would imply $[\![\Delta\Delta]\!] \cdot a \sqsubseteq [\![\Delta\Delta]\!]$ against the hypothesis. Consider the function

$$f(z) = \begin{cases} [\![\Delta\Delta]\!] & \text{if } \ z \not\sqsubseteq [\![\Delta\Delta]\!] \\ \perp & \text{otherwise.} \end{cases}$$

Obviously f is a continuous function. Then

$$\begin{aligned}[\![X]\!] \cdot f \cdot a &= [\![\Delta\Delta]\!] \cdot (f \cdot ([\![\Delta\Delta]\!] \cdot a) \cdot [\![\Delta\Delta]\!])(f \cdot [\![\Delta\Delta]\!] \cdot ([\![\Delta\Delta]\!] \cdot a)) \\ &= [\![\Delta\Delta]\!] \cdot ([\![\Delta\Delta]\!] \cdot [\![\Delta\Delta]\!]) \cdot \perp = [\![\Delta\Delta]\!] \cdot \perp \qquad \text{(by ii)}.\end{aligned}$$

while

$$\begin{aligned}[\![Z]\!] \cdot f \cdot a &= [\![\Delta\Delta]\!] \cdot (f([\![\Delta\Delta]\!] \cdot a)([\![\Delta\Delta]\!] \cdot a)) = [\![\Delta\Delta]\!] \cdot ([\![\Delta\Delta]\!] \cdot ([\![\Delta\Delta]\!] \cdot a)) \\ &= [\![\Delta\Delta]\!] \cdot a.\end{aligned}$$

This last contradiction proves the theorem. ▌

This last theorem has a remarkable corollary.

COROLLARY 4. *The ccc of CPOs and Scott continuous functions is incomplete with respect to λ-calculus.*

The incompleteness is due to the fact that the set of all continuous functions is too rich. Obviously, none of the functions f defined in the previous theorem is λ-definable. This phenomenon seems connected with the fact that the class of continuous functions also includes non-sequential functions while λ-calculus is sequential in nature. This result should be compared with the one of Plotkin [9, pp. 234–237], which shows that the full structure of continuous functions does not provide a fully abstract semantics for LCF, unless new additional parallel features are introduced in the language. Corollary 4 should also be compared to the result obtained in [5] concerning the denotational semantics of Landin's SECD machine.

Several natural questions now arise:

(i) Is there a satisfactory characterization of λ-theories induced by topological models?

(ii) Why is the operational behaviour of certain terms sensitive to unstable or non-sequential functions? Do certain terms have a parallel "flavour?"

(iii) is there a notion of topological submodel of a topological λ-model with respect to which completeness for λ-calculus could be proved?

The authors will deal with these problems in a future paper.

Returning to the theory $\mathscr{C}_{\Lambda^0}$ in itself, it is interesting to contrast it with $\mathscr{H}^*$. While closed terms are never equated to open terms in $\mathscr{C}_{\Lambda^0}$, in $\mathscr{H}^*$ every closed term can be equated to an open term. In fact, let M be a closed term, and let M' be obtained from M substituting each variable x with the term $(Y_z Fx)$ ($Y_z F$ is an open term equal to I in $\mathscr{H}^*$); then $M' = M$ and obviously M' is an open term. Moreover $\mathscr{C}_{\Lambda^0}$ discriminates closed terms equal in $\mathscr{H}^*$. Consider, for example, the fixed point combinator Y and the closed term Y^2 defined in Section 3, Remark 2. While $\mathscr{H}^* \vdash Y = Y^2$, $\mathscr{C}_{\Lambda^0} \vdash Y \neq Y^2$; take the context

$$C[\ ] = [\ ](\lambda x_1 x_2 x_3 . x_1(x_3 t) O) \qquad (\text{where } O = \lambda xy.y);$$

It is easy to verify that $C[Y^2]$ reduces to a closed term, while $C[Y]$ does not.

Let us examine now the theory of the model $\mathscr{T}$, which we call $\mathscr{C}^{\mathscr{T}}$. Since closed terms belonging to Λ_I have special properties in $\mathscr{C}^{\mathscr{T}}$, as proved in the preceding section, it is natural to compare $\mathscr{C}^{\mathscr{T}}$ with the contextual theory obtained by choosing as the set of observables the set Λ_I^0.

The result of this comparison is given by the following:

THEOREM 12. $\mathscr{C}^{\mathscr{T}} \subsetneq \mathscr{C}_{\Lambda_I^0}$.

To prove this theorem, we need to prove the following lemma.

LEMMA 14. *Let $M \in \Lambda^*$ be a nf, but $M \notin \Lambda_I^*$. Then, if ξ is the environment such that $\forall x\ \xi(x) = \Phi$, $[\![M]\!]_\xi^{\mathscr{T}} \not\sqsupseteq \Phi$.*

Proof. $M \notin \Lambda_I^*$ implies that there is at least a subterm of M which does not belong to Λ_I^*. We prove the lemma by induction on the construction of the approximant M, starting from one of the innermost subterms of M not belonging to Λ_I^*. Let N be such a subterm; clearly $N = \lambda x.P$, and $x \notin FV(P)$:

$$[\![N]\!]_\xi^{\mathscr{T}} \sqsupseteq \Phi \Rightarrow [\![N]\!]_\xi^{\mathscr{T}} \sqsupseteq f_{\Phi,\Phi} \qquad \text{and}$$

$$[\![N]\!]_\xi^{\mathscr{T}} \sqsupseteq f_{\top_0,\top_0} \Rightarrow [\![N]\!]_\xi^{\mathscr{T}} . \top_0 \sqsupseteq \top_0 \Rightarrow [\![P]\!]_{\xi[x/\top_0]}^{\mathscr{T}} \sqsupseteq \top_0$$

$$\Rightarrow (\text{since } x \notin FV(P)) \qquad [\![P]\!]_{\xi[x/\Phi]}^{\mathscr{T}} \sqsupseteq \top_0$$

which is impossible by Lemma 11. The induction step is similar to the proof of Lemma 12, since, in $\mathscr{T}$ as in $\mathscr{P}$, $[\![\Phi A_1 \ldots A_m]\!]_\xi^{\mathscr{T}} \sqsupseteq \Phi$ iff $\forall i\ [\![A_i]\!]_\xi^{\mathscr{T}} \sqsupseteq \Phi$. ∎

Proof of Theorem 12. As in the proof of Theorem 9, we prove that in $\mathscr{C}^{\mathscr{T}}$ a term belonging to Λ_I^0 cannot be equated to a term which does not belong to Λ_I^0. Obviously, the following property holds:

PROPERTY 5. *$M \notin \Lambda_I^0$; then $\forall A \in \mathcal{A}(M)$, $A \notin \bar{\Lambda}_I^{*0}$.*

Now, let M be a term, not belonging to Λ_I^0. Then, by Lemma 14, there is an environment ξ such that $\forall A \in \mathcal{A}(M)$, $[\![A]\!]_\xi^{\mathcal{T}} \not\sqsupseteq \Phi$; then $[\![M]\!]_\xi^{\mathcal{T}} \sqsupseteq \Gamma \Phi$. Since, by Property 3, $M \in \Lambda_I^0$ implies $[\![M]\!]^{\mathcal{T}} \sqsupseteq \Phi$, the inclusion is proved.

To prove that the inclusion is proper, consider the two terms M and N defined in Theorem 5. Remember that $\mathcal{T} \models M \neq N$. It is easy to check that $\mathscr{C}_{\Lambda_I^0} \vdash M = N$, since there is not a term L such that $\Delta\Delta(\lambda x.L(L(\Delta\Delta))) \in \Lambda_I^0 (\notin \Lambda_I^0)$ and $\Delta\Delta(\lambda x.L) \notin \Lambda_I^0 (\in \Lambda_I^0)$.

Remark 4. Note that $\mathscr{C}^{\mathcal{T}}$ can equate open and closed terms. In fact it is easy to check that $\mathscr{C} \models Y_z K = YK$, where Y_z is defined in Remark 1.

Remark 5. With respect to the possibility of modelling $\mathscr{C}_{\Lambda_I^0}$ with a topological model, $\mathscr{C}_{\Lambda_I^0}$ is similar to $\mathscr{C}_{\Lambda^0}$. Namely if D is a topological model, $\mathscr{C}^D \subseteq \mathscr{C}_{\Lambda_I^0}$, then $\mathscr{C}^D \subsetneq \mathscr{C}_{\Lambda_I^0}$.

6. REFINEMENTS OF THE APPROXIMATION THEOREM

In this section we study three $D_{(i,j)}$ models, namely $\mathscr{S}$, $\mathscr{V}$, and the standard D_∞ [10], defined as follows:

DEFINITION 13. (i) Let

$$D^0 = \begin{cases} \top_0 \\ \psi \\ \bot_0 \end{cases}$$

and let

$$\begin{aligned} i_s(\bot_0) &= C_{\bot_0}, & i_v(\bot_0) &= C_{\bot_0} \\ i_s(\psi) &= f_{\psi,\psi}, & i_v(\psi) &= f_{\psi,\psi} \\ i_s(\top_0) &= f_{\psi,\top_0}, & i_v(\top_0) &= f_{\bot_0,\top_0} \end{aligned}$$

$$\mathscr{S} = D_{(i_s, j_s)}$$
$$\mathscr{V} = D_{(i_v, j_v)}.$$

(ii) Let

$$D^0 = \begin{cases} \top_0 \\ \bot_0 \end{cases}$$

and let

$$i_0(\bot_0) = C_{\bot_0}$$
$$i_0(\top_0) = f_{\bot_0,\top_0}$$
$$D_\infty = D_{(i_0, j_0)}.$$

Let D be the either $\mathscr{S}$ or $\mathscr{V}$ or D_∞. The general approximation theorem is unsatisfactory for D, since the constant Φ is not λ-definable. Therefore we need a refinement of the approximation theorem which yields λ-definable approximants. To prove that Φ is not λ-definable, we need a deeper analysis of the behaviour of 0-projections.

Let Λ^{**} be the language obtained from Λ adjoining the new constant χ. The intended interpretation of χ in D is either $[\![\chi]\!]_\xi^{\mathscr{S}} = \psi$ or $[\![\chi]\!]_\xi^{\mathscr{V}} = \psi$ or $[\![\chi]\!]_\xi^{D_\infty} = \bot_0$ according to which D is considered.

PROPERTY 6. In D

— $[\![\chi]\!]^D \subsetneq [\![\Phi]\!]^D = ([\![I]\!]^D)_1$.
— $[\![(\lambda x.P)^0 Q]\!]_\xi^D = [\![(\lambda x.P)^0 (\chi Q)]\!]_\xi^D \sqsubseteq [\![P^0[x/\chi Q]]\!]_\xi^D$.

Proof. A direct case analysis yields the result. ▌

COROLLARY 5. *In D, Φ is not λ-definable.*

Proof. It is sufficient to prove that there is no unsolvable term U such that $[\![U]\!]_\xi^D = [\![\Phi]\!]^D = ([\![I]\!]^D)_1$. From Lemma 4 we have that $[\![U]\!]_\xi^D = \bigsqcup \{[\![W^I]\!]_\xi^D \mid U \twoheadrightarrow_\beta U' \wedge (U', J)$ is an indexed term $\wedge (W, I)$ is the wnf of $(U', J)\}$. By Property 6, since U is unsolvable, any $[\![W^I]\!]_\xi^D \sqsubseteq [\![\lambda x_1 \ldots x_n.\chi]\!]_\xi^D$. But $\forall n\, [\![\lambda x_1 \ldots x_n.\chi]\!]_\xi^D \not\sqsupseteq [\![\Phi]\!]^D$. Φ being compact, we have $[\![U]\!]_\xi^D \neq [\![\Phi]\!]_\xi^D$. ▌

We now define a new set of approximants.

DEFINITION 14. Let $M \in \Lambda$.

(i) The direct χ-approximant of M is the nf $A \in \Lambda^{**}$ obtained from M by replacing each redex $(\lambda x.P)Q$ by $\chi(\lambda x.P)Q$.

(ii) The set of χ-approximants of M is the set $\mathscr{A}_\chi(M) = \{A \mid \exists M'\ M \twoheadrightarrow_\beta M'$ and A is the direct χ-approximant of $M'\}$.

THEOREM 13. $M \in \Lambda$.

$$[\![M]\!]_\xi^D = \bigsqcup \{[\![A]\!]_\xi^D \mid A \in \mathscr{A}_\chi(M)\}.$$

Proof. Let $A_\chi(M)$ and $A_\Phi(M)$ be respectively the direct χ-approximant and the direct approximant of M. Clearly $[\![A_\chi(M)]\!]_\xi^D \sqsubseteq [\![A_\Phi(M)]\!]_\xi^D$. Let M' be the term obtained from M by a complete reduction with respect to all redexes in M. $[\![A_\Phi(M)]\!]_\xi^D \sqsubseteq [\![A_\chi(M')]\!]_\xi^D$; in fact, by Property 5, $[\![\Phi(\lambda x.P)Q]\!]_\xi^D \sqsubseteq [\![P[x/\chi Q]]\!]_\xi^D \sqsubseteq [\![P']\!]_\xi^D$, where P' is obtained from $P[x/\chi Q]$ by deleting the constant χ when Q is not the operator of a redex, and $A_\chi(M')$ is obtained from $A_\Phi(M)$ by repeatedly substituting expressions of the form $\Phi(\lambda x.P)Q$ with P' as above. ▌

Remark 6. It is easy to verify that Theorem 13, in the case $D = D_\infty$, gives exactly the direct approximants defined in [12].

The rest of the section is devoted to the analysis of $\mathscr{S}$ and $\mathscr{V}$ by means of χ-approximants. First of all, we note that, χ being the interpretation of all closed unsolvable terms of $\beta-\eta$-order 0, we succeed in giving λ-definable approximants.

From now on let D be either $\mathscr{S}$ or $\mathscr{V}$. We prove that D is has a semisensible theory, properly included in $\mathscr{C}_{\Lambda^0}$, hence not maximal. It should be interesting to compare $\mathscr{C}^D$ with $\mathscr{C}_{S^0}$, where S^0 is the set of closed solvable terms.

LEMMA 15. *$M \in \Lambda$. If ξ is the environment such that $\forall x\, \xi(x) \sqsupseteq \psi$ then $[\![M]\!]^D_\xi \sqsupseteq \psi$.*

Proof. By induction on the structure of M. ∎

LEMMA 16. *Let $A \in \Lambda^{**}$ be an nf, and let $FV(A) \neq \varnothing$. Then for every environment ξ such that $\exists x \in FV(A)$, $\xi(x) = \perp$ and $\forall y \not\equiv x\, \xi(y) = \psi$, $[\![A]\!]^D_\xi \not\sqsupseteq \psi$.*

Proof. By induction on the structure of A.

$A \equiv x$. Obvious.

$A \equiv \lambda x.A'$. Since $FV(A) \neq \varnothing$, there is at least one variable $y \neq x$ which occurs free in A'. Let ξ be an environment satisfying the conditions for A, with $\xi(y) = \perp$. By the induction hypothesis, $[\![A']\!]^D_{\xi[x/\chi]} \not\sqsupseteq \psi$ and, since

$$[\![\lambda x.A']\!]^D_\xi \cdot \psi = [\![A']\!]^D_{\xi[x/\psi]} \not\sqsupseteq \psi, \qquad [\![A]\!]^D_\xi \not\sqsupseteq \psi.$$

$A \equiv \chi A_1 \ldots A_m$. For every ξ satisfying the conditions w.r.t. A, $\exists j\, [\![A_j]\!]^D_\xi \not\sqsupseteq \psi$. Hence the thesis follows by the induction hypothesis.

$A \equiv xA_1 \ldots A_m$. Every ξ satisfying the hypothesis for A is such that $\xi(x)$ can be either $\perp$ or ψ. In the first case the proof is obvious, in the second one is similar to the preceding one. ∎

Then, since Property 4 obviously holds, the very same argument used in Section 5, in order to prove the Theorem 9, can be used to prove the following:

THEOREM 14. *$\mathscr{C}^D \subseteq \mathscr{C}_{\Lambda^0}$.*

THEOREM 15. *$\mathscr{C}^D$ is semisensible.*

Proof. Let $M = \lambda x_1 \ldots x_n.\zeta M_1 \ldots M_m$ be any solvable term, where ζ is a variable, either free or bound, and let ξ be the environment such that $\forall x\, \xi(x) = \top_0$.

Then $[\![M]\!]^D_\xi \underbrace{\top_0 \ldots \top_0}_{n} = [\![\zeta M_1 \ldots M_m]\!]^D_{\xi[x_i/\top_0]} = \top_0$, using Lemma 15; so M is such that, for any integer $t \geq n$, $[\![M]\!]^D_\xi \underbrace{\top_0 \ldots \top_0}_{t} = \top_0$.

Let U be an unsolvable term. If $[\![U]\!]^D_\xi = [\![M]\!]^D_\xi$, $\forall t \geq n$

$$[\![U]\!]^D_\xi \top_0 \ldots \top_0 = \bigsqcup \{[\![A]\!]^D_\xi \underbrace{\top_0 \ldots \top_0} \mid A \in \mathscr{A}_\chi(U)\} = \top_0,$$

and, since $\top_0$ is a compact element, this implies that $\exists A \in \mathcal{A}_\chi(U)$ such that $[\![A]\!]_\xi^D \underbrace{\top_0 \ldots \top_0}_{t} = \top_0$, $\forall t \geqslant n$.

But this is impossible, since every $A \in \mathcal{A}_\chi(U)$ is such that there exists an integer p s.t. $[\![A]\!]_\xi^D \sqsubseteq [\![\lambda x_1 \ldots x_p . \chi]\!]$ (since, from the approximation theorem, every approximant of an unsolvable term is a χ-head nf). ∎

COROLLARY 6. $\mathscr{C}^D \subsetneq \mathscr{C}_{\Lambda^0}$.

Note that, since $\mathscr{C}^D$, as $\mathscr{C}^{\mathscr{P}}$, does not equate open and closed terms, $\mathscr{C}^D \vdash Y \neq Y_z$ (see Remark 1). Then $\mathscr{C}^D$ is an example of a semisensible theory in which terms with the same Böhm trees are not equated.

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